REASONABLE MODEL OF THE UNIVERSE

by

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An insistence upon realism and simplicity is used along with Einstein's theory of gravitation to show that already ascertained facts on the large scale structure of the universe lead inexorably to a particular cosmological model. This is homogeneous and isotropic with negative curvature. The model demonstrates that the universe is infinite and currently expanding from a big bang in the past, the variation in cosmic scale with time being almost sinusoidal, so that eventually the universe will recondense and give rise to an ensuing big bang.

The properties of the model may be elicited both in terms of the elementary functions and as expansions in a parameter $\alpha \approx 0{\cdot}0075$ which characterizes its matter and radiation densities. According to the model, the epoch of equal densities occurred after $6{\cdot}6 \times 10^6$ years, the present age of the universe is $1{\cdot}0 \times 10^{10}$ years, and the duration of its current cycle will be $4{\cdot}0 \times 10^{10}$ years. The luminosity distance for the model is exhibited in an instructive form.

§1. Introduction

The prevailing models of the material universe are disparate in several of its crucial aspects. Yet there is available a framework of established data——a framework consisting of theoretical principles and cosmical observations——that has withstood amendment for so long as to render it very persuasive. By resolving now to work in conformity with these data (of which an account is given in §2), and using them as criteria for the admissibility of plausible models, it will be found possible to settle the qualitative aspects of the universe and to limit its quantitative aspects.

The 'admissible' models of the cosmical structure will be determined in §4. Each of them will be capable of describing the universe over its whole duration, as all of them comprise a mixture of its two main components - ponderable matter and radiation. Models containing more components were examined by Vajk (1969), and interactions between the components were included by McIntosh (1968), Knight & Bergmann (1974) and May (1975), but the effects produced by such elaborations are quite small.

Our reasons for preferring a particular model will not all have become apparent until §6, where we shall be able to identify it. It will be distinguished from the models of other authors in §8.

§2. Theoretical equations and observational values

In this section we assemble the standard equations and values that will delineate the admissible models. These values were drawn from the treatise by Weinberg (1972).

2(a) *The Robertson-Walker line element*

Observation supports the cosmological tenet that the field of receding galaxies may be modelled by a distribution that is smoothed out and uniform (i.e. homogeneous and isotropic). A uniform model will have a metric of the form

$$ds^2 = dt^2 - S^2(t)\left[\frac{dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)}{(1 + \tfrac{1}{4}kr^2)^2}\right], \tag{2.1}$$

taking the velocity of light to be unity. We shall need to determine the curvature index k (= -1, 0 or 1) and the cosmic scale factor $S(t)$.

We can compare the measured red shift z (= $\Delta\lambda/\lambda$) and apparent magnitude of a receding galaxy by employing its luminosity distance from us as derived from (2.1). For a galaxy with $z \ll 1$, this distance has the Taylor expansion

$$d_L = (z/H_0)[1 + \tfrac{1}{2}(1 - q_0)z + O(z^2)], \tag{2.2}$$

where the Hubble constant H_0 and the deceleration parameter q_0 denote the present values $H(t_0)$ and $q(t_0)$ of the functions

$$H(t) = \dot{S}/S, \tag{2.3}$$

$$q(t) = -S\ddot{S}/\dot{S}^2 = -\tfrac{1}{2}S\, d(\ell n\, \dot{S}^2)/dS. \tag{2.4}$$

It has been found that the values

$$H_0^{-1} = 1{\cdot}3 \times 10^{10}\ \text{yrs}, \tag{2.5}$$

$$q_0 \approx 1, \tag{2.6}$$

accord satisfactorily with the measurements.

2(b) *The Einstein gravitation equations*

The cosmical evolution is ruled by gravitation, for which the most general acceptable equations are

$$R^{\mu}_{\nu} - \tfrac{1}{2} R\delta^{\mu}_{\nu} + \Lambda\delta^{\mu}_{\nu} = -\kappa T^{\mu}_{\nu} .$$

The cosmological constant Λ, which ought not to be at variance with the observed motions of the planets, was estimated by Tolman (1934, p.474) to lie in the range

$$-2 \times 10^{-18} < \Lambda < 5{\cdot}7 \times 10^{-18} \text{ yrs}^{-2} . \tag{2.7}$$

The gravitation equations, on being applied to the metric (2.1), establish that uniform models are filled with perfect fluid, whose density ρ and pressure p depend only on the cosmic time t. The gravitation equations thereby reduce to

$$\dot{S}^2 = \tfrac{1}{3} (\kappa\rho + \Lambda) S^2 - k \tag{2.8}$$

and to a second order differential equation. Uniform models containing distinct components of matter and radiation have

$$\rho = \rho_{mat} + \rho_{rad} , \tag{2.9}$$

with $p = p_{rad} = \frac{1}{3} \rho_{rad}$, since p_{mat} is unimportant. For such models, the second order equation leads to the relations

$$\tfrac{1}{3} \kappa\rho_{mat} = M/S^3 , \qquad \tfrac{1}{3} \kappa\rho_{rad} = N/S^4 , \tag{2.10}$$

where M and N are constants. We see that the radiation component will predominate when S is small.

Abundant observational evidence indicates that the prime source of matter is the galaxies, estimated to contribute a smoothed-out density of

$$(\rho_{mat})_0 = 3\cdot1 \times 10^{-31}\ \mathrm{g\,cm^{-3}}\ . \qquad (2.11)$$

The natural microwave background of photons contributes a much smaller density of $4\cdot4 \times 10^{-34}\ \mathrm{g\,cm^{-3}}$, so that, in the knowledge that neutrinos and gravitons are expected to contribute a comparable density, a *token* estimate for the collective radiation density of the order of

$$(\rho_{rad})_0 = 1 \times 10^{-33}\ \mathrm{g\,cm^{-3}} \qquad (2.12)$$

seems reasonable. (The densities observed to arise from other possible sources are negligible by comparison.)

§3. Preliminary neglect of matter and radiation

The sparsity of these contents of the universe suggests that our enlightenment may be served by considering the outcome of neglecting their *present* action upon the metric. The irrefutable logic of elementary mathematics will show us that there is only one empty uniform model which is, for the present era, expanding and concordant with the observed values of H_0 and q_0. (This inadequate model will constitute the limit as $\rho \to 0$ of the realistic models in §4.)

The differential equation (2.8) with $\rho = 0$ is

$$\dot{S}^2 = \tfrac{1}{3}\Lambda S^2 - k\ , \qquad (3.1)$$

in which $\dot{S} > 0$ during the present expansion. So it is that, with $S(t_0) = S_0$, the definitions (2.3) and (2.4) yield

$$H_0^2 S_0^2 = \tfrac{1}{3}\Lambda S_0^2 - k , \qquad \tfrac{1}{3}\Lambda S_0^2 = kq_0/(1+q_0) ,$$

implying that neither Λ nor k can be zero. By eliminating Λ, we get

$$H_0^2 S_0^2 = -k/(1+q_0) ,$$

and therefore $k = -1$ and $S_0 = H_0^{-1}(1+q_0)^{-\frac{1}{2}}$. The cosmological constant is thus given by $\Lambda = -3H_0^2 q_0$.

If we were also to elect, rather more immoderately, to disregard the effects of matter and radiation at all other times as well, the model would then be defined by the solution of (3.1) which vanishes at $t = 0$, namely

$$S(t) = h^{-1} \sin ht ,$$

where $h = H_0 q_0^{\frac{1}{2}}$. (Cf. equation (6.8) of Robertson 1933.) The definitions (2.3) and (2.4) would accordingly assume the explicit forms

$$H/h = \cot ht , \qquad q = \tan^2 ht .$$

Hence it would follow that the age to the present epoch is

$$t_0 = h^{-1} \operatorname{arccot}(H_0/h) = h^{-1} \arcsin[q_0/(1+q_0)]^{\frac{1}{2}} ,$$

and the age to the epoch with $q = 1$ is

$$t_1 = h^{-1}\pi/4 .$$

We note for use in the appendix that, since $q_0 \approx 1$, these epochs are evidently in close proximity.

For this elementary model, the exact luminosity distance is given by (7.13).

§4. Realistic models with matter and radiation

In order to study models of the universe that contain both matter and radiation, we substitute (2.9) and (2.10) into the differential equation (2.8) to get

$$\dot{S}^2 = \tfrac{1}{3}\Lambda S^2 - k + \frac{M}{S} + \frac{N}{S^2} \equiv \mathcal{S}(S) . \tag{4.1}$$

This central equation has been classified by, amongst others, Harrison (1967), Stabell (1968) and Payne (1970). It gives, using (2.3) and (2.4) again,

$$H^2 = \mathcal{S}(S)/S^2 , \tag{4.2}$$

$$q = -\tfrac{1}{2} S\mathcal{S}'(S)/\mathcal{S}(S) . \tag{4.3}$$

Further, if the epoch at which $q = 1$ be still denoted by t_1, and $S(t_1) = S_1$, we have from (4.3) that

$$\tfrac{1}{3}\Lambda S_1^2 = \tfrac{1}{4}(2k - M/S_1) , \tag{4.4}$$

being the relation we shall use to eliminate Λ.

Let us now define a *cosmic scale variable* u by

$$S(t) = uS_1 , \tag{4.5}$$

so that $q = 1$ when $u = 1$. And let us introduce the abbreviations

$$P = M/S_1 \quad \text{(matter)} , \qquad Q = N/S_1^2 \quad \text{(radiation)} .$$

Then the differential equation (4.1) becomes, with the help of (4.4),

$$\dot{S}^2 = -f(u)/u^2 , \tag{4.6}$$

depending on the *crucial quartic*

$$f(u) = \tfrac{1}{4}(P - 2k)u^4 + ku^2 - Pu - Q ,$$

with the derivative

$$f'(u) = (P - 2k)(u - 1)[u^2 + u + P/(P - 2k)] . \qquad (4.7)$$

In the appendix it is proved from (4.1), using the values quoted in §2, that $k = -1$; and thus all the realistic models are *hyperbolic*. The behaviour of any one of these models, being governed by the equation (4.6), and hence by the quartic f, will be sensitive to that model's precise values of P and Q. As a typical special case, to provide definiteness for our analysis of f, we shall select the values

$$P = 0{\cdot}015 , \qquad Q = 0{\cdot}00005 , \qquad (4.8)$$

as estimated approximately in the appendix from the data in §2. (Our final model will not be restricted to the use of (4.8).)

The functions f and q, for $k = -1$ and these values of P and Q, are plotted as functions of u in Figure 1.

Figure 1 to appear here.

§5. Roots of the typical quartic equation

The physical behaviour associated with the typical model (4.8) is determined by the quartic

$$f(u) = \tfrac{1}{2}(1+\tfrac{1}{2}P)u^4 - u^2 - Pu - Q\,. \qquad (5.1)$$

We shall therefore have need of the turning points of this quartic and the roots of its equation $f = 0$.

The derivative (4.7), which is affected solely by the matter, shows that f has *turning points* at $u = 1$ and at both the solutions of

$$u^2 + u + P/(2+P) = 0\,.$$

Now as P is small, these solutions are approximately equal to -1 and 0; but a better estimate for the small solution is

$$-\tfrac{1}{2}P + O(P^3) = -0{\cdot}00750\,,$$

thus demonstrating that the middle turning point has $u < 0$, a fact which is barely discernible in Figure 1.

The *roots of the quartic equation* $f = 0$ may first be estimated in a rough manner by simply ignoring P and Q as we did in §3. This gives the equation $(\tfrac{1}{2}u^2 - 1)u^2 = 0$, and hence the roots $-\sqrt{2}$, 0, 0, $+\sqrt{2}$.

To estimate the two small roots of $f = 0$ more accurately, we note that u^4 will be very small for these roots, so that one may determine them to a sufficiently good approximation from the truncated equation

$$u^2 + Pu + Q = 0\,.$$

When real, the roots of this equation are negative, and are given by

$$-\tfrac{1}{2}P \pm (\tfrac{1}{4}P^2 - Q)^{\frac{1}{2}}\,. \qquad (5.2)$$

The two small roots of $f = 0$ are thus

$$- 0{\cdot}00750 \pm 0{\cdot}00250 .$$

The other two roots of $f = 0$ are

$$- 1{\cdot}40140 , \qquad + 1{\cdot}41640 .$$

The sum of the four roots vanishes because there is no u^3 term in f.

According to the differential equation (4.6), with the present typical f, the universe will first expand from its initial singular state at $u = 0$ until it reaches its maximum size at $u = 1{\cdot}41640$; at this epoch $\dot{S}$ will change sign, and the universe will then recondense to a further singular state with $u = 0$.

Were we to solve this differential equation for this typical model, or for any other realistic model in which the two small roots *differ*, the solution would involve an elliptic integral. If, however, it were arranged for the two small roots to be equal, the evaluation of the integral could be completed in terms of the elementary functions, and thence would arise a more tractable solution. We shall therefore proceed to show that f can be simplified in this way without changing it significantly in the physical range $0 \leqslant u \leqslant 1{\cdot}41640$.

First we ask: Would the quartic then still conform with the data? This may immediately be answered by inspecting (5.2). We see that the *approximate* condition for the two small roots of $f = 0$ to coincide is that Q should satisfy

$$Q = \tfrac{1}{4} P^2 = 0{\cdot}00005625 .$$

(Here we have decided to retain the same value of P because it is more reliable.) This revised value of Q corresponds to a radiation density of about $1{\cdot}2 \times 10^{-33}$ g cm^{-3}; this is, of course, equally as acceptable as the previous token estimate of 1×10^{-33} g cm^{-3}.

§6. The simplest realistic model

One is accordingly justified in dispensing with the *general* quartic (5.1) that applies to all the models typified by (4.8). In place of that quartic, one can use a *modified* quartic

$$\hat{f}(u) = \tfrac{1}{2}(1 + \tfrac{1}{2}\hat{P})u^4 - u^2 - \hat{P}u - \hat{Q}\,, \tag{6.1}$$

which shares the same general form but which possesses a repeated small root. Therefore it may also be expressed in the form

$$\hat{f}(u) = K(u+\alpha)^2(u-\beta)(u-2\alpha+\beta)\,, \tag{6.2}$$

in which K, α and β are positive constants.

This alternative form shows that $\hat{f}$ vanishes at $u = \beta$: it follows that if we choose $\beta = 1{\cdot}41640$ (the previous positive root), u will have the same physical range as before. Furthermore, the form (6.1) implies that $\hat{f}'(1) = 0$: hence the turning point in the physical range will remain at $u = 1$.

The coefficients of the same powers of u in (6.1) and (6.2) can be equated. This establishes that (6.2) must have, in fact, the more specific form

$$\hat{f}(u) = \tfrac{1}{2}(1-\alpha+\alpha^2)^{-1}(u+\alpha)^2[(u-\alpha)^2 - 2(1-\alpha)]\,, \tag{6.3}$$

involving a single constant α, whose value is fixed in terms of the chosen β by

$$\alpha = \beta - 1 - (3-2\beta)^{\frac{1}{2}} = 0{\cdot}00750\,.$$

The other quantities may now be defined in terms of α; we find

$$\beta = \alpha + [2(1-\alpha)]^{\frac{1}{2}}, \tag{6.4}$$

and

$$\hat{P} = 2\alpha(1-\alpha)(1-\alpha+\alpha^2)^{-1} \approx 2\alpha ,$$

$$\hat{Q} = \alpha^2(1-\alpha-\tfrac{1}{2}\alpha^2)(1-\alpha+\alpha^2)^{-1} \approx \alpha^2 ,$$

showing that $\hat{Q} \approx \frac{1}{4}\hat{P}^2$. For $\alpha = 0{\cdot}0075$ exactly, these definitions give, to four significant figures,

$$\hat{P} = 0{\cdot}01500 , \qquad \hat{Q} = 0{\cdot}00005625 .$$

To summarize §§4,5 and 6: The *simplest model* among the realistic models with matter and radiation is the one having the quartic $\hat{f}$ defined by (6.3), in which α serves as a small positive parameter. The observational values in §2 lead to $\alpha = 0{\cdot}0075$.

§7. Some properties of the model

The differential equation governing the evolution of the α-model of §6 is, from (4.6),

$$\dot{S} = \pm[-\hat{f}(u)]^{\frac{1}{2}}/u , \tag{7.1}$$

in which (+) is associated with expansion and (-) with contraction. It has been so modified as to be soluble in terms of the elementary functions, and therefore we shall easily be able to determine the ages of interest and the luminosity distance.

Let us first write some useful equations. For the present epoch, our definition (4.5) reads

$$S(t_0) = S_0 = u_0 S_1 , \tag{7.2}$$

so that (4.2) gives

$$H_0^2 S_0^2 = -\hat{f}(u_0)/u_0^2 , \tag{7.3}$$

while (4.3) gives

$$q_0 = -[\tfrac{1}{2}(1+\tfrac{1}{2}\hat{P})u_0^4 + \tfrac{1}{2}\hat{P}u_0 + \hat{Q}]/\hat{f}(u_0) .$$

This equation admits of a solution for u_0 in terms of q_0 and α; it is, to an adequate approximation,

$$u_0 = \mu[1 + \nu\alpha + O(\alpha^2)] , \tag{7.4}$$

where

$$\mu = [2q_0/(1+q_0)]^{\frac{1}{2}} , \qquad \nu = \mu^{-1}[1 - 1/(2q_0)] - \tfrac{1}{2} .$$

To facilitate the evaluation of the integrals, we shall employ the substitution

$$v = \arcsin\{(u-\alpha)/[2(1-\alpha)]^{\frac{1}{2}}\} ,$$

and for brevity put

$$\alpha/[2(1-\alpha)]^{\frac{1}{2}} = \varepsilon , \qquad \arcsin 2\varepsilon = \psi ,$$

so that $\psi < \frac{1}{2}\pi$ for $\alpha < \frac{1}{2}$. The value of v when $t = 0$ and $u = 0$ is $v_i = -\arcsin \varepsilon$.

7(a) *Age to the epoch of equal densities*

For the α-model, (2.10) and (4.5) lead to

$$\rho_{rad}/\rho_{mat} = \hat{Q}/(\hat{P}u) , \tag{7.5}$$

and thus $\rho_{rad} = \rho_{mat}$ at that epoch for which

$$u = u_e = \hat{Q}/\hat{P} = \tfrac{1}{2}\alpha(1-\alpha-\tfrac{1}{2}\alpha^2)/(1-\alpha) = \tfrac{1}{2}\alpha[1+O(\alpha^2)] .$$

The age of the universe at that time was

$$t_e = \int_0^{S_e} \frac{dS}{\dot{S}} = S_1 \int_0^{u_e} u[-\hat{f}(u)]^{-\frac{1}{2}} du ,$$

using (4.5) and (7.1). So, with help from (7.2) and (7.3), we obtain

$$t_e = (C/H_0)[v_e - v_i - \varepsilon \int_{v_i}^{v_e} \frac{dv}{2\varepsilon + \sin v}]$$

$$= (C/H_0)[v_e - v_i - \tfrac{1}{2}\tan\psi \; L(v_i, v_e, \psi)] , \tag{7.6}$$

where $L(v_i, v_e, \psi) = \ell n \dfrac{\sin\frac{1}{2}(v_e+\psi)\cos\frac{1}{2}(v_i-\psi)}{\sin\frac{1}{2}(v_i+\psi)\cos\frac{1}{2}(v_e-\psi)}$,

and $C = [2(1-\alpha+\alpha^2)]^{\frac{1}{2}}[-\hat{f}(u_0)]^{\frac{1}{2}}/u_0^2$

$$= q_0^{-\frac{1}{2}}[1+\mu^{-3}\alpha + O(\alpha^2)] ,$$

with additional help from (7.4). In the result (7.6), the quantities v_i, v_e and ψ depend on α alone, whereas the coefficient C depends on q_0 also.

As $q_0 \approx 1$, we shall estimate the ages by taking $q_0 = 1$. For $\alpha = 0{\cdot}0075$, our result (7.6) gives

$$t_e|_{q_0=1} = 0{\cdot}000507\, H_0^{-1} = 6{\cdot}6 \times 10^6 \text{ yrs} ,$$

using the observational value (2.5).

7(b) *Age to the present epoch*

The present age is easily seen to be

$$t_0 = (C/H_0)[v_0 - v_i - \tfrac{1}{2}\tan\psi\, L(v_i, v_0, \psi)] ,$$

where v_0 may be found with the aid of (7.4). The age is calculable from this exact result, or from its approximation

$$t_0\big|_{q_0=1} = [\tfrac{1}{4}\pi + 0{\cdot}707\,\alpha\,\ln\alpha + 0{\cdot}880\,\alpha + O(\alpha^2)]\,H_0^{-1} .$$

Either method yields, for $\alpha = 0{\cdot}0075$,

$$t_0\big|_{q_0=1} = 0{\cdot}766\,H_0^{-1} = 1{\cdot}0 \times 10^{10}\ \text{yrs.}$$

7(c) *Age to the epoch of maximum size*

The universe reaches its maximum size when u attains its maximum value according to (6.4), i.e.

$$u_m = \beta = \alpha + [2(1-\alpha)]^{\frac{1}{2}} ,$$

so that $v_m = \tfrac{1}{2}\pi$. The age at this epoch can thus be expressed as

$$t_m = (C/H_0)[\tfrac{1}{2}\pi - v_i - \tfrac{1}{2}\tan\psi\, L(v_i, \tfrac{1}{2}\pi, \psi)] .$$

For $\alpha = 0{\cdot}0075$, this gives $t_m\big|_{q_0=1} = 1{\cdot}556\,H_0^{-1}$, and hence the duration of a complete cycle is

$$2\,t_m\big|_{q_0=1} = 4{\cdot}0 \times 10^{10}\ \text{yrs.}$$

7(d) *Luminosity distance for the model*

For any uniform model (2.1) with $k = -1$, the luminosity distance from the observer ($r = 0$) to a source ($r = r_s$) having red shift z is

$$d_L = (1+z)\,\frac{r_s}{1 - \tfrac{1}{4}r_s^{\,2}}\,S_0 . \qquad (7.7)$$

We shall evaluate this with the aid of another distance indicator which is defined by

$$\ell_s = \int_0^{r_s} \frac{dr}{1 - \frac{1}{4} r^2} \; . \tag{7.8}$$

First, we have

$$\frac{r_s}{1 - \frac{1}{4} r_s^{\,2}} = \sinh \ell_s \; . \tag{7.9}$$

We may also substitute for the right hand member in (7.8) an equivalent member which follows from the equation of a radial null geodesic. Applying this procedure to the α-model shows that

$$\ell_s = \int_{t_s}^{t_0} \frac{dt}{S} = \int_{S_s}^{S_0} \frac{dS}{S\dot{S}} = \int_{u_s}^{u_0} [-\hat{f}(u)]^{-\frac{1}{2}}\, du \; ,$$

where in the last integral, using $1 + z = S_0/S_s$, we have $u_s = u_0/(1+z)$. We thus get, after integrating,

$$\ell_s = \left[\frac{1 - \alpha + \alpha^2}{(1+\alpha)(1-2\alpha)}\right]^{\frac{1}{2}} L(v_s, v_0, \psi) \; . \tag{7.10}$$

With the help of (7.3) and (7.4) we also have

$$S_0 = H_0^{-1}(1+q_0)^{-\frac{1}{2}}\,[1 + \tfrac{3}{2}\mu^{-1}\alpha + O(\alpha^2)] \; . \tag{7.11}$$

Hence, by substitution into (7.7), from (7.9), (7.10) and (7.11), we now obtain the luminosity distance as a function of H_0, q_0, α and z. This function can most usefully be written as an expansion in α, of the form

$$d_L = D(z)[1 + E(z)\alpha + O(\alpha^2)] \; . \tag{7.12}$$

The first factor, which is defined by

$$D(z) = H_0^{-1} q_0^{-1} (1+z)^2 \{[1 + q_0 - q_0/(1+z)^2]^{\frac{1}{2}} - 1\} \; , \tag{7.13}$$

is identical with the exact luminosity distance for the empty model of §3. The influence of matter and radiation gives rise to those terms coming after unity in the second factor.

For galaxies with $z \ll 1$, (7.12) may be expanded as a series in z, to give

$$d_L = (z/H_0)[1 + \tfrac{1}{2}(1 - q_0)z + \tfrac{1}{2}(1 + q_0)(q_0 - 2\mu^{-1}\alpha)z^2 + O(z^3) + O(\alpha^2)] .$$

(This is an extension of (2.2).) But it is inevitable that αz^2 will be undetectably small, so we cannot use this to determine α from observation, as is done for H_0 and q_0.

To determine α from (7.12), very remote sources with large values of z will need to be taken into account, and the exact expression for E(z) used. This is

$$E(z) = \{E_1 + E_2 + (E_1 + E_3)[1 + q_0 - q_0/(1 + z)^2]^{-\frac{1}{2}}\}/(q_0\mu) ,$$

where $E_1 = [q_0 - 2 + 2/(1 + q_0)](1 + z)^2/(2 + z)$,

$$E_2 = \tfrac{1}{2}(1 - q_0) - q_0\mu^2 z ,$$

$$E_3 = \tfrac{1}{2}(3q_0 - 1) - q_0[1 - 2/(1 + q_0)]z - q_0\mu^2/(1 + z) .$$

In particular,

$$E(z)\big|_{q_0 = 1} = z[(1 + 4z + 2z^2)^{-\frac{1}{2}} - 1] .$$

§8. Conclusion

The α-model of the universe studied in §§ 6 and 7 is the simplest one of the allowed models, which are those conforming with the data in §2. Owing to the smallness of α, this canonical model does not differ much from the elementary 0-model examined in §3 (i.e. the model with $k = -1$ and $S = h^{-1}|\sin ht|$ for all t). Thus the α-model is hyperbolic and possesses an infinite spatial extent (so we are not facing our own backs). Also its cosmological constant $\Lambda = -3(H_0/C)^2 < 0$, causing the α-model to oscillate between recurring condensed states——these being ultimately singular. The duration of the current cycle of the α-model has been found to be $4{\cdot}0 \times 10^{10}$ years. Because the present age of the universe is a quarter of this period, the stars and galaxies have had time to form.

After the parameter α ($\approx 0{\cdot}0075$) has been adjusted in accordance with the observed density of matter, a decisive test of the α-model is the tenability of (7.5) in its present form, viz

$$(\rho_{rad}/\rho_{mat})_0 = \hat{Q}/(\hat{P}u_0) \approx \tfrac{1}{2}\alpha/u_0 \approx \tfrac{1}{2}\alpha[(1+q_0)/(2q_0)]^{\frac{1}{2}}.$$

This requires the ratio of the present densities to be about $0{\cdot}004$.

We shall close by giving a brief account of the earlier work. The uniform line element (2.1) was originally derived by Friedmann (1922) for the spherical case $k = 1$. Models containing both matter and radiation, with the density relations (2.10) holding, were then discussed by Lemaître (1927) for this same case. (The case studied by Knight & Bergmann (1974) and Coquereaux & Grossmann (1982).) Analytical solutions for the case $k = -1$ were first obtained by Alpher & Herman (1949): but they chose $\Lambda = 0$. This choice has since been adopted by most other authors, such as Chernin (1965), Chiu (1967), Cohen (1967), McIntosh (1968), Vajk (1969), Sapar (1970), and eventually by May (1975). The general classification scheme of Robertson (1933) does cover models with $k = -1$ and $\Lambda < 0$, and so do the schemes of Harrison (1967), Stabell (1968) and Payne (1970). Models of this kind have also been considered in connexion with gravitational radiation (see, for example, Waylen 1978 and Zimmerman & Hellings 1980).

So far as I am aware from the literature that is available here, there has been no previous study of the α-model.

§9. Appendix

We shall use the observational values given in §2 to determine the value of k and to obtain approximate estimates of P and Q.

At the epoch t_1, (4.2) and (4.4) lead to

$$\tfrac{1}{2}\frac{k}{S_1{}^2} = \frac{3}{4}\frac{P}{S_1{}^2} + \frac{Q}{S_1{}^2} - H_1{}^2 , \tag{9.1}$$

while the relations (2.10) become

$$\tfrac{1}{3}\kappa(\rho_{mat})_1 = P/S_1{}^2 , \qquad \tfrac{1}{3}\kappa(\rho_{rad})_1 = Q/S_1{}^2 . \tag{9.2}$$

Now $q = 1$ at t_1 and (2.6) asserts that $q_0 \approx 1$ at t_0, so it is possible to infer that t_1 is moderately near to t_0 (see Figure 1). The values of H, ρ_{mat}, ρ_{rad} at t_1 are thus furnished with reasonable accuracy by the corresponding values (2.5), (2.11), (2.12) at t_0 (see the curves (a), (b), (c) in Figure 1). Consequently we have

$$H_1{}^2 = 59 \times 10^{-22} \text{ yrs}^{-2} ,$$

and, using (9.2) with $\kappa = 1{\cdot}670 \times 10^9 \text{ cm}^3 \text{ g}^{-1} \text{ yrs}^{-2}$,

$$\left.\begin{aligned} P/S_1{}^2 &= 1{\cdot}7 \times 10^{-22} \\ Q/S_1{}^2 &= 0{\cdot}006 \times 10^{-22} \end{aligned}\right\} \text{yrs}^{-2} . \qquad (9.3)$$

Substituting these values into (9.1), we get

$$\tfrac{1}{2}k/S_1{}^2 = (1{\cdot}3 + 0{\cdot}006 - 59) \times 10^{-22} = -58 \times 10^{-22} \text{ yrs}^{-2} ,$$

using two significant figures. It follows that we must have $k = -1$ and, with somewhat lesser certitude, $S_1{}^2 = 86 \times 10^{18}$ yrs^2. From (9.3) we now obtain the values

$$P = 0{\cdot}015 , \qquad Q = 0{\cdot}00005 ,$$

which are those selected at (4.8).

The cosmological constant can be found from (4.4): thus

$$\Lambda = -\tfrac{3}{2}(1 + \tfrac{1}{2}P)/S_1{}^2 = -1{\cdot}8 \times 10^{-20} \text{ yrs}^{-2} .$$

This is just 1% of Tolman's permissible lower limit (2.7).

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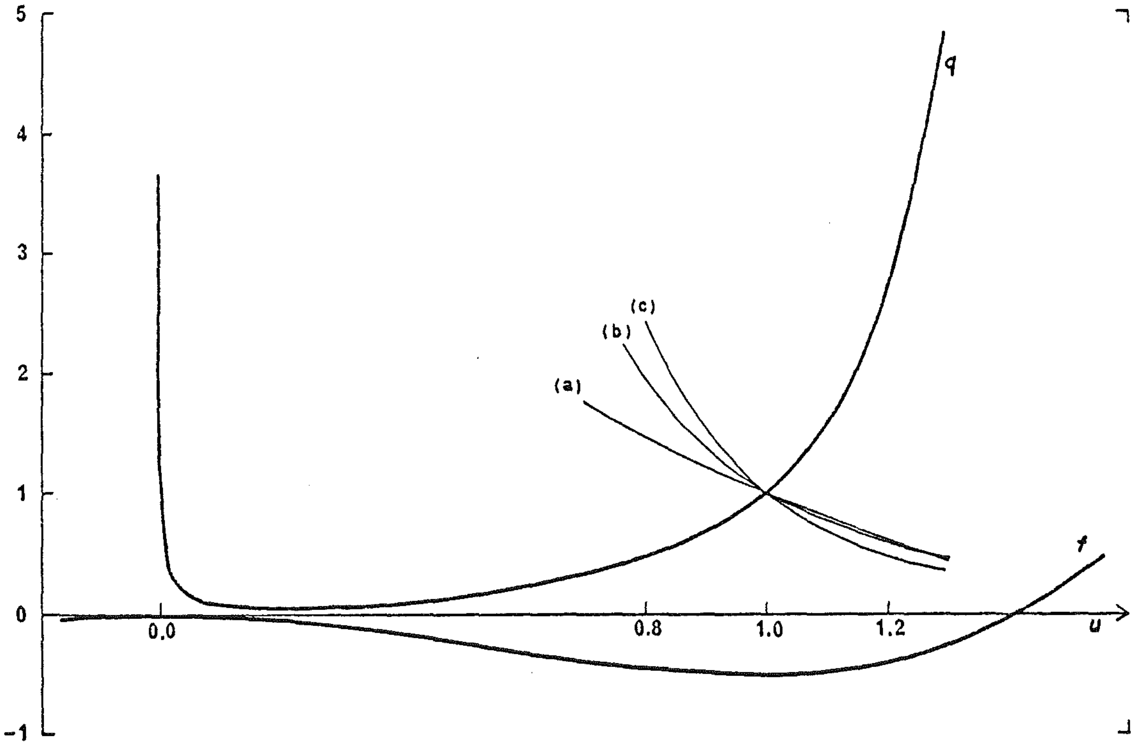

Figure 1. The functions f and q shown for a domain of u that includes its physical range. Since the model contains radiation, both $f = -Q < 0$ and $q = 1$ when $u = 0$. In addition $q = 1$ when $u = 1$. (Vertical asymptotes for q occur where $f = 0$.) The curves (a),(b),(c) depict the respective ratios H/H_1, $\rho_{mat}/(\rho_{mat})_1$, $\rho_{rad}/(\rho_{rad})_1$.

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